Schur Polynomials

classical def. of Schur poly $s_\lambda(x_1, \dots, x_n)$

fix n and $\lambda = (\lambda_1, \dots, \lambda_n)$ partition

$$\lambda_1 \geq \dots \geq \lambda_n$$

for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$

$$a_\alpha(x_1, \dots, x_n) := \det \begin{bmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \vdots & & & \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \cdots & x_n^{\alpha_n} \end{bmatrix}$$

$$= \det \left(x_j^{\alpha_i} \right)_{1 \leq i, j \leq n}$$

$$= \sum_{w = (w_1 \cdots w_n) \in S_n} \operatorname{sign}(w) \cdot w(x^\alpha)$$

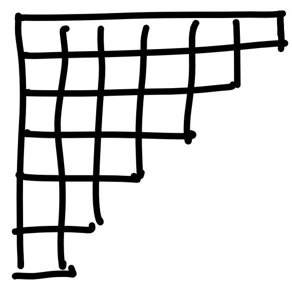
where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

and $w(x^\alpha) = x_{w_1}^{\alpha_1} \cdots x_{w_n}^{\alpha_n}$

$$= \left(\sum_{w \in S_n} \mathrm{sgn}(w)\, w \right)(x^\alpha)$$

In this way, this $\uparrow$ operator is antisymmetrises

$$\delta := (n-1, n-2, \dots, 1, 0) =$$



then $a_\delta(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j)$

<u>Def</u>: $s_\lambda(x_1, \dots, x_n) := \dfrac{s_{\lambda+\delta}(x_1, \dots, x_n)}{s_\delta(x_1, \dots, x_n)}$

Properties:

- s_λ is symmetric polynomial.
- s_λ has positive integer coefficients.

this is apparent from Comb. def. of s_λ.

Schubert Polynomials

$\mathfrak{S}_w(x_1, \dots, x_n)$ where $w \in S_n$ is a permut.

not symmetric.

classical def:

(Right) weak Bruhat order on S_n.

Example: $(n=3)$

$$\begin{tikzcd}
 & w_0 = 321 \arrow[dl, dash, "s_1"'] \arrow[dr, dash, "s_2"] & \\
s_1 s_2 = 231 \arrow[d, dash, "s_2"] & & 312 = s_2 s_1 \arrow[d, dash, "s_1"] \\
s_1 = 213 \arrow[dr, dash, "s_1"'] & & 132 = s_2 \arrow[dl, dash, "s_2"] \\
 & 123 &
\end{tikzcd}$$

conventions for permutations:

$$w = w_1 \cdots w_n$$
$$= \begin{pmatrix} 1 & \cdots & n \\ w_1 & \cdots & w_n \end{pmatrix} \quad \text{so } w : k \to w_k$$

Covering relations for (right) wk Bruhat ord.

- $w s_i \gtrdot w$ and length increases

or

- switch $w = \ldots w_i \, w_{i+1} \ldots$ if $w_i < w_{i+1}$

Divided difference operation

$$\partial_i : \mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}[x_1, \ldots, x_n] \quad 1 \leq i \leq n-1$$

$$f(x_1, \ldots, x_n) \longrightarrow \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i+1}\, x_i \cdots x_n)}{x_i - x_{i+1}}$$

$$= \frac{f - s_i f}{x_i - x_{i+1}}$$

or just write $\dfrac{1}{x_i - x_{i+1}}(1 - s_i)$.

Classical Definition of Schubert Polynomials

① $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 = x^\delta$

② $\mathfrak{S}_w = \partial_i(\mathfrak{S}_{ws_i})$ if $w < ws_i$ in wk Bruhat

Lemma: $\mathfrak{S}_w$ is well defined.

Example:

$$\begin{tikzcd}[column sep=small]
 & \mathfrak{S}_{321} = x_1^2 x_2 \arrow[dl, dash, "\partial_1"'] \arrow[dr, dash, "\partial_2"] & \\
\mathfrak{S}_{231} = x_1 x_2 \arrow[d, dash, "\partial_2"'] & & x_1^2 = \mathfrak{S}_{312} \arrow[d, dash, "\partial_1"] \\
\mathfrak{S}_{213} = x_1 \arrow[dr, dash] & & x_1 + x_2 = \mathfrak{S}_{132} \arrow[dl, dash] \\
 & 1 = \mathfrak{S}_{123} &
\end{tikzcd}$$

- $\mathfrak{S}_w$ has positive integer coefficients.

Schur $\quad S_\lambda = \dfrac{1}{\prod_{i<j}(x_i - x_j)} \left(\sum_w \operatorname{sgn}(w) w \right) (x^{\lambda+\delta})$

vs

Schubert $\quad \mathfrak{S}_\lambda = \dots \dfrac{1}{x_i - x_{i+1}} (1 - s_i)(x_i^\delta)$

<u>Next Lecture:</u> Schur are special cases of Schubert.